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FLEXURE OF DOUBLE CANTILEVER BEAMS

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STRUCTURAL DIVISION

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PAPERS

FLEXURE OF DOUBLE CANTILEVER BEAMS

By F. E. Wolosewick,[1] M. ASCE

Synopsis

A cantilever beam is a simple structural element. When two cantilever beams at right angles to each other are joined rigidly, a complicated structure results. Each cantilever beam is subject to torsion and bending.

Construction of this type is becoming more frequent in buildings, water works, power plants, and industrial works. The elimination of the corner caisson and the supporting columns is an invitation to economy. Frequently such supporting members cannot be provided because of physical restrictions and architectural requirements.

Since concrete is weak in torsion, and existing steel shapes are unsuitable for resisting high torsional stresses, it is important to determine the correct magnitudes of tosional moments in such supporting members.

Conventional supporting members have average height-to-depth ratios. For such sections the elastic functions, as defined by A, B, and C, are small. Consequently, torsional moments will be high and must require careful consideration. As these elastic functions increase numerically, torsional moments have secondary effects.

The resulting equations have been arranged in such a manner that for any loading conditions, evaluation of forces, can be accomplished by direct substitution and by simultaneous solutions of necessary equations.

Introduction

Two cantilever beams, at right angles to each other, having their free ends connected either rigidly, or by a universal joint, form a double cantilever system. Such systems are shown isometrically in Fig. 1. Under normal construction procedure, either in concrete or steel, ends C would be joined

Note.—Written comments are invited for publication; the last discussion should be submitted by May 1, 1953.

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together. When so constructed, resistance to rotation would be developed by both beams. Torsional moments would then be transferred from one beam to the other. When rotations would be allowed at joint C, torsional moments would not exist, and both cantilevers would be subject to bending moments.

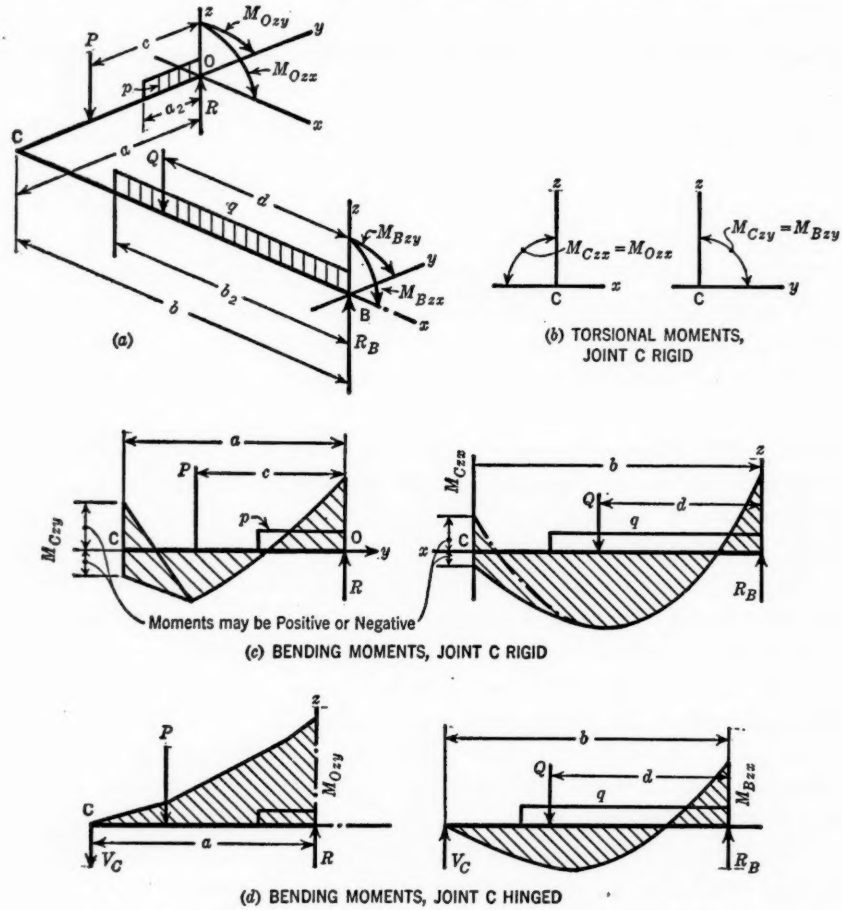

FIG. 1.—DOUBLE CANTILEVERS WITH FIXED AND RIGID JUNCTION POINTS

In the analysis and discussions which follow, point O is assumed as the origin. The reaction at point O is called R, and is positive if acting as shown in Fig. 1. The bending moment M_{Ozy} is positive when rotating clockwise in Fig. 1(a). Torsional moment M_{Ozx} is considered positive if acting as shown in Fig. 1(a). It may have zero value or be negative, depending on the ratios of lengths, elastic and torsional properties of the cantilever beams defined by functions A, B, and C, and the magnitudes of applied loads. In discussions of right-angle cantilevers, the observer is assumed to be standing at four o'clock, and looking toward point O.

All moments are defined by subscripts designating the planes in which they rotate. A subscript preceding a plane designation would define the point where rotations are occurring. This designation is easier to visualize than rotations about an axis.

With the forces at the origin known, the bending and torsional moments can be determined at any point in both spans by statics. Torsional moments do not enter into the equation for flexure on any point in member a, and must therefore be treated independently. Similarly, bending moments are not affected by torsional moments, and as such must be treated independently. This would be true of course, as long as both cantilevers are at right angles to each other. With an angle other than 90°, components of torsional and bending moments must be considered in determining the resultant effect. At point C the torsional moment of one span becomes the bending moment for the other span, at right angles to the one considered. Likewise at point C, the bending moment of one span becomes the torsional moment for the other span, at right angles to the one considered.

General Case of Continuity

The most general case of a system of restraining forces is shown in Fig. 1(a). There are six restraining forces in the system, three at point O and three at point B. Since three forces located correctly preserve equilibrium, the remaining unknowns must be determined from continuity. These unknowns can be obtained most conveniently by the theory of least work. Special cases frequently occur in which it is inconvenient or impossible to

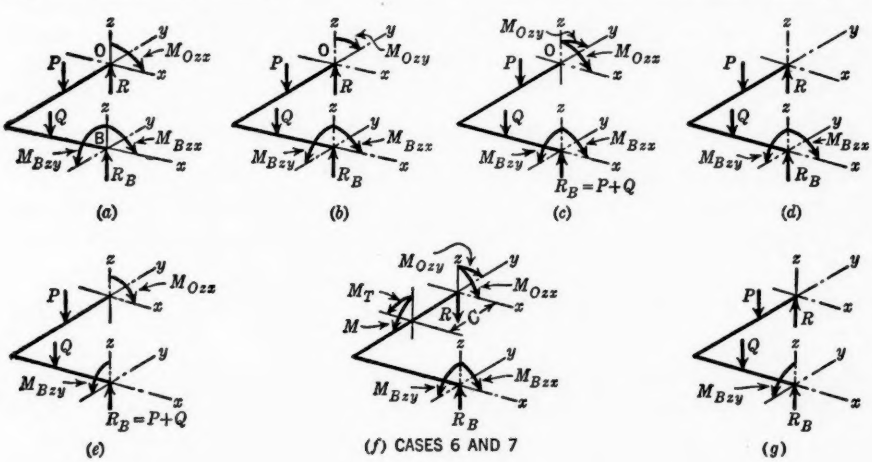

Fig. 2.

develop all restraining forces at point O or point B. These cases are shown isometrically in Fig. 2.

Thus in Fig. 2(a), there are two restraining forces at point O, (M_{Ozx} and R) and in Fig. 2(b) there are two forces—M_{Ozy} and R at point O. Fig. 2(c) is a special case of a guided cantilever at point O, with restraining moments

M_{Ozx} and M_{Ozy}, without reaction R. Fig. 2(d) is a special case of double cantilever, where resisting moments cannot be developed, and only reaction R is available.

Further specialized cases of a double cantilever can be developed. Thus at point O either one of the two moments can be acting without the benefit of reaction R. All such cases of guided cantilevers are of limited usefulness structurally, but may have considerable applications in instrumentation engineering. Fig. 2(g) shows an unstable structure. Although three forces are shown acting, which numerically satisfy the equation of stability, a little study will show instability. Thus, it is not possible to have two reactions, and a single moment to preserve equilibrium. Equilibrium can only be preserved when one of the reactions is replaced by a moment, rotating in the plane at 90° to the other moment.

This brief summary indicates that location of restraining moments and reactions can be made dependent on the physical conditions affecting the design.

Development of General Equations

The most general case of a system of restraining forces at the origin is shown in Fig. 1(a), with three forces at points O and B. Such a system is indeterminate to the third degree. For any manner of loading on spans a or b, the equation expressing work for the entire system is

$$W = \int \frac{M^2\, ds}{2\, E\, I} + \int \frac{M^2\, ds}{2\, G\, J} \quad \text{(1)}$$

in which M is the moment acting at any distance from point O; ds is a small increment of distance; E is the modulus of elasticity; I is the rectangular moment of inertia of a cantilever; G is the torsional modulus of elasticity $= \frac{0.5\, E}{1 + \mu}$; J is the torsional moment of inertia of a cantilever and μ is Poisson's ratio (0.25 for steel and 1/7 for concrete).

To determine the unknown forces, three differentiations are required, one with respect to R, then with respect to M_{Ozx}, and finally with respect to M_{Ozy}; thus,

$$\frac{\partial W}{\partial R} = \int M \frac{\partial M}{\partial R} \frac{ds}{E\, I} + \int M \frac{\partial M}{\partial R} \frac{ds}{G\, J} = 0 \quad \text{(2a)}$$

$$\frac{\partial W}{\partial M_{Ozx}} = \int M \frac{\partial M}{\partial M_{Ozx}} \frac{ds}{E\, I} + \int M \frac{\partial M}{\partial M_{Ozx}} \frac{ds}{G\, J} = 0 \quad \text{(2b)}$$

$$\frac{\partial W}{\partial M_{Ozy}} = \int M \frac{\partial M}{\partial M_{Ozy}} \frac{ds}{E\, I} + \int M \frac{\partial M}{\partial M_{Ozy}} \frac{ds}{G\, J} = 0 \quad \text{(2c)}$$

Integrations of Eqs. 2 lead to solutions listed as Eqs. 3 to 17, in Table 1. These equations contain three unknown forces which can be evaluated algebraically. The formal solutions of Eqs. 2 have been deleted from this paper because of their length. For certain loading conditions and boundary conditions they become somewhat unmanageable.

Particular Solutions

From general solutions for any loading conditions, a series of special equations can be developed, dependent solely on the boundary conditions at points O and B.

As a specific illustration, assume that M_{Ozx} cannot be developed, but that R and M_{Ozy} can be resisted. To determine forces from this boundary condition, use equations a and c in Table 1, with M_{Ozx} being set equal to zero. Simultaneous solutions will determine the unknown forces.

When both moments at point O cannot be resisted, but only the reaction R is available to resist vertical shears, its magnitude can be determined from Eq. c of any loading condition in Table 1. This equation is the only one containing values of M_{Ozx} and M_{Ozy}, both of which can be equated to zero and still offer a solution for R. For a condition of a guided cantilever, reaction R is zero, and moments M_{Ozx} and M_{Ozy} are assumed to be acting. The values of these moments can be determined by combining Eqs. a (Table 1) with Eqs. b and solving simultaneously with Eqs. c, with R in all three being set to zero. A special case of a guided cantilever would be one in which only moment M_{Ozy}

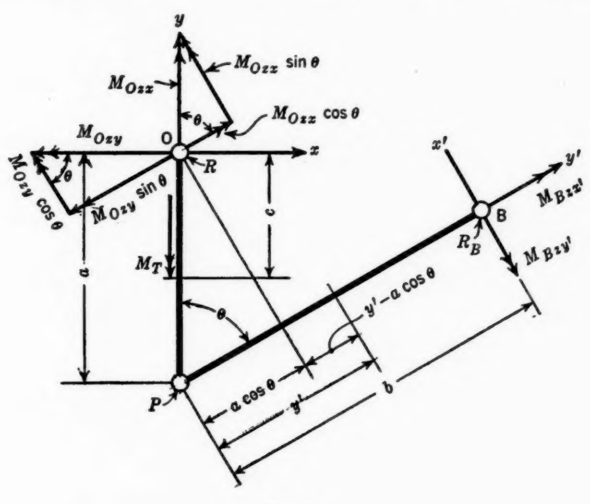

Fig. 3

is acting. The value of this moment can be determined from Eqs. a, with R being zero. Similarly when only M_{Ozx} is required, Eqs. b will furnish the required answer.

General Case for a Variable Angle

Two cases will be evaluated for this condition, one for a uniform load on both spans, the other for a torque M_T acting at a distance of c from point O.

With restraining forces at point O, the cantilever of length b at an angle θ will be acted by components of bending and torsional moments, to be combined vectorially. These are shown in Fig. 3.

TABLE 1.—EQUATIONS FOR VARIOUS LOADING CONDITIONS [a]

	EQ. 3—CONCENTRATED LOAD P ON SPAN a
a	$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = P\,a\,(\gamma - \tfrac{1}{2}\,\gamma^2 - \tfrac{1}{2} - r\,A + \gamma\,r\,A)$
b	$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = -\,P\,a\,\frac{r^2}{2}$
c	$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} + r\,A\right) = P\,a\left(\frac{1}{3} + \frac{\gamma^3}{6} - \frac{\gamma}{2} + \frac{i\,r^3}{3} + r\,A - \gamma\,r\,A\right)$
	EQ. 4—CONCENTRATED LOAD Q ON SPAN b
a	$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = 0$
b	$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = -\,Q\,\frac{a\,\delta^2}{2}$
c	$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} + r\,A\right) = Q\,a\,i\,r^3\left(\frac{1}{3} - \frac{(1-\delta)}{2} + \frac{(1-\delta)^3}{6}\right)$
	EQ. 5—UNIFORM LOAD p OVER ENTIRE LENGTH a AND UNIFORM LOAD q OVER ENTIRE LENGTH b
a	$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = -\,p\,a^2\left(\frac{1}{6} + \frac{r\,A}{2}\right)$
b	$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = -\,p\,a^2\left(\frac{r^2}{2} + \frac{m\,r^3}{6}\right)$
c	$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} + r\,A\right) = p\,a^2\left(\frac{1}{8} + \frac{i\,r^3}{3} + \frac{i\,m\,r^4}{8} + \frac{r\,A}{2}\right)$
	EQ. 6—UNIFORM LOAD p OVER ENTIRE LENGTH a ONLY
a	$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = -\,p\,a^2\left(\frac{1}{6} + \frac{r\,A}{2}\right)$
b	$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = -\,p\,a^2\,\frac{r^2}{2}$
c	$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} + r\,A\right) = p\,a^2\left(\frac{1}{8} + \frac{i\,r^3}{3} + \frac{r\,A}{2}\right)$
	EQ. 7—UNIFORM LOAD q OVER ENTIRE LENGTH b ONLY
a	$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = 0.$
b	$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = -\,q\,\frac{b^2\,r}{6}$
c	$-M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} + r\,A\right) = q\,b^2\,\frac{i\,r^2}{8}$

[a] Equation numbering is in sequence with the numbering in the text.

In the vectorial system, sense and direction of moments can be determined readily by visualizing vectors as arrows. If one were to grasp the arrow with the right hand (the thumb pointing toward the head of the arrow), the direction in which the fingers wrap themselves around the shaft will indicate the direction in which the moments rotate. The right-hand system of rotations is used throughout. General equations for moments are not stated in this paper because of their length. The pattern in general is the same as for Eq. 1. Partial differentials are similar to Eqs. 2. Angle θ is constant. Final solutions are shown in Table 1.

TABLE 1.—(*Continued*)

Eq. 8—Torsional Moment M_T Acting at Any Point, Distance c from Point O

a	$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = 0$
b	$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = M_T\,(1 - \gamma + r)$
c	$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} + r\,A\right) = M_T\,\frac{i\,r^2}{2}$

Eq. 9—Bending Moment M Acting at Any Distance c from Point O

a	$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = M\,(1 - \gamma + r\,A)$
b	$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = 0$
c	$M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} + r\,A\right) = M\left(\frac{1}{2} - \frac{\gamma^2}{2} + r\,A\right)$

Eq. 10—Uniform Load q over Distance b_2. Length b_2 Measured from Point B to End of Load

a	$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = 0$
b	$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = -\,q\,\frac{b^2\,r}{2}\,[\tfrac{1}{3} - (1 - \beta_2) + \tfrac{2}{3}\,(1 - \beta_2)^3 + (1 - \beta_2)^2\,\beta_2]$
c	$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} + r\,A\right)$
	$= q\,b^2\,i\,r^2\left[\frac{1}{8} - \frac{1}{3}\,(1 - \beta_2) + \frac{1}{4}\,(1 - \beta_2)^2 - \frac{1}{24}\,(1 - \beta_2)^4\right]$

Eq. 11—Uniform Load p over Distance a_2. Length a_2 Measured from Point O

a	$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = -\,p\,a^2\left[\frac{1}{6}\,\alpha^3_2 - \frac{\alpha^2_2}{2} + r\,A\left(\alpha_2 - \frac{\alpha^2_2}{2}\right) + \frac{\alpha_2}{2}\right]$
b	$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = -\,p\,a^2\,\frac{r^2\,\alpha_2}{2}$
c	$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} + r\,A\right)$
	$= p\,a^2\left\{\frac{\alpha^4_2}{8} + \alpha_2\left[\frac{(1 - \alpha^3_2)}{3} - \frac{\alpha_2(1 - \alpha^2_2)^2}{4} + \frac{(2 - \alpha_2)\,r\,A}{2} + \frac{r^3\,i}{3}\right]\right\}$

Eq. 12—Forces and Moments May Be Determined When Support O Settles or Rotates

a	$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = \Theta_{zy}\,E\,I_a$
b	$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = \Theta_{zx}\,E\,I_a$
c	$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} + r\,A\right) = \Delta Z\,E\,I_a$

Eq. 13—Uniform Load p over Spans a and b Taking Shear Distortion into Consideration

a	$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = -\,p\,a^2\left(\frac{1}{6} + \frac{r\,A}{2}\right)$
b	$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = -\,p\,a^2\left(\frac{r^2}{2} + \frac{r^3}{6}\right)$
c	$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left[\frac{1}{3} + \frac{i\,r^3}{3} + r\,A + \frac{E\,I_a}{a^2\,G\,S_a}\left(1 + r\,\frac{S_a}{S_b}\right)\right]$
	$= p\,a^2\left[\frac{1}{8} + \frac{i\,r^3}{3} + \frac{1}{8}\,i\,r^4 + \frac{r\,A}{2} + \frac{E\,I_a}{a^2\,G\,S_a}\left(1 + r\,\frac{S_a}{S_b}\right)\right]$

TABLE 1.—(*Continued*)

Eq. 14—Load P at Distance a from Point O Taking Shear Distortions into Consideration

a
$$M_{Ozy}\,(1 + r\,A) - R\,a\,(\tfrac{1}{2} + r\,A) = 0$$

b
$$M_{Ozx}\,(r + B) - R\,a\,\frac{r^2}{2} = -\,P\,a\,\frac{r^2}{2}$$

c
$$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,A) - M_{Ozx}\,\frac{i\,r^2}{2} + R\,a\left[\frac{1}{3} + \frac{i\,r^3}{3} + r\,A + \underline{\frac{E\,I_a}{G\,S_a\,a^2}\left(1 + r\,\frac{S_a}{S_b}\right)}\right] = P\,a\left(\frac{i\,r^3}{3} + r\,\underline{\frac{E\,I_a}{a^2\,G\,S_b}}\right)$$

Eq. 15—Uniform Load p on Spans a and b for Any Variable Angle θ Between Cantilevers

a
$$M_{Ozy}\,(1 + i\,r\cos^2\theta + r\,A\sin^2\theta) + M_{Ozx}\left(\frac{i\,r\sin 2\,\theta}{2} - \frac{r\,A\sin 2\,\theta}{2}\right) + R\,a\left(\frac{i\,r^2\cos\theta}{2} - i\,r\cos^2\theta\right.$$
$$\left. -\,\frac{1}{2} - r\,A\sin^2\theta\right) = -\,p\,a^2\left(\frac{1}{6} - \frac{i\,r^2\cos\theta}{2} + \frac{i\,r\cos^2\theta}{2} - \frac{i\,r^3\cos\theta}{6} + \frac{r\,A\sin^2\theta}{2}\right)$$

b
$$M_{Ozy}\,(\tfrac{1}{2}\,r\,i\sin 2\,\theta - \tfrac{1}{2}\,r\,A\sin 2\,\theta) + M_{Ozx}\,(C + r\,i\sin^2\theta + r\,A\cos^2\theta) + R\,a\,(\tfrac{1}{2}\,r^2\,i\sin\theta$$
$$-\,\tfrac{1}{2}\,r\,i\sin 2\,\theta + \tfrac{1}{2}\,r\,A\sin 2\,\theta) = p\,a^2\left(\frac{1}{2}\,r^2\,i\sin\theta - \frac{1}{4}\,r\,i\sin 2\,\theta + \frac{r^3\,i\sin\theta}{6} + \frac{1}{4}\,r\,A\sin 2\,\theta\right)$$

c
$$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,i\cos^2\theta - \tfrac{1}{2}\,r^2\,i\cos\theta + r\,A\sin^2\theta) + M_{Ozx}\left(\frac{r^2\,i\sin\theta}{2}\right.$$
$$\left. -\,\frac{1}{2}\,r\,i\sin 2\,\theta + \frac{1}{2}\,r\,A\sin 2\,\theta\right) + R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} - r^2\,i\cos\theta + r\,i\cos^2\theta + r\,A\sin^2\theta\right)$$
$$= p\,a^2\left(\frac{1}{8} + \frac{1}{3}\,i\,r^3 - \frac{1}{2}\,r^2\,i\cos\theta - \frac{r^2\,i\cos\theta}{4} + \frac{r\,i\cos^2\theta}{2} + \frac{r^4\,i}{8} - \frac{r^3\,i\cos\theta}{6} + \frac{r\,A\sin^2\theta}{2}\right)$$

Eq. 16—Concentrated Load P at Distance a from Point O for Any Variable Angle θ Between Cantilevers

a
$$M_{Ozy}\,(1 + i\,r\cos^2\theta + r\,A\sin^2\theta) + M_{Ozx}\left(\frac{i\,r\sin 2\,\theta}{2} - \frac{r\,A\sin 2\,\theta}{2}\right)$$
$$+\,R\,a\left(\frac{i\,r^2\cos\theta}{2} - i\,r\cos^2\theta - \frac{1}{2} - r\,A\sin^2\theta\right) = \tfrac{1}{2}\,P\,a\,i\,r^2\cos\theta$$

b
$$M_{Ozy}\,(\tfrac{1}{2}\,r\,i\sin 2\,\theta - \tfrac{1}{2}\,r\,A\sin 2\,\theta) + M_{Ozx}\,(C + r\,i\sin^2\theta + r\,A\cos^2\theta)$$
$$+\,R\,a\,(\tfrac{1}{2}\,r^2\,i\sin\theta - \tfrac{1}{2}\,r\,i\sin 2\,\theta + \tfrac{1}{2}\,r\,A\sin 2\,\theta) = \tfrac{1}{2}\,P\,a\,i\,r^2\sin\theta$$

c
$$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,i\cos^2\theta - \tfrac{1}{2}\,r^2\,i\cos\theta + r\,A\sin^2\theta) + M_{Ozx}\left(\frac{r^2\,i\sin\theta}{2} - \frac{1}{2}\,r\,i\sin 2\,\theta + \frac{1}{2}\,r\,A\sin 2\,\theta\right)$$
$$+\,R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} - r^2\,i\cos\theta + r\,i\cos^2\theta + r\,A\sin^2\theta\right) = P\,a\left(\frac{i\,r^3}{3} - \frac{r^2}{2}\cos\theta\right)$$

Eq. 17—Torsional Moment M_T Acting at Distance c from Point O on Span a for Any Variable Angle θ Between Cantilevers

a
$$M_{Ozy}\,(1 + i\,r\cos^2\theta + r\,A\sin^2\theta) + M_{Ozx}\left(\frac{i\,r\sin 2\,\theta}{2} - \frac{r\,A\sin 2\,\theta}{2}\right)$$
$$+\,R\,a\left(\frac{i\,r^2\cos\theta}{2} - i\,r\cos^2\theta - \frac{1}{2} - r\,A\sin^2\theta\right) = M_T\,(\tfrac{1}{2}\,i\,r\sin 2\,\theta - \tfrac{1}{2}\,r\,A\sin 2\,\theta)$$

b
$$M_{Ozy}\,(\tfrac{1}{2}\,r\,i\sin 2\,\theta - \tfrac{1}{2}\,r\,A\sin 2\,\theta) + M_{Ozx}\,(C + r\,i\sin^2\theta + r\,A\cos^2\theta)$$
$$+\,R\,a\,(\tfrac{1}{2}\,r^2\,i\sin\theta - \tfrac{1}{2}\,r\,i\sin 2\,\theta + \tfrac{1}{2}\,r\,A\sin 2\,\theta) = M_T\,(C - \gamma\,C + i\,r\sin^2\theta + r\,A\cos^2\theta)$$

c
$$-\,M_{Ozy}\,(\tfrac{1}{2} + r\,i\cos^2\theta - \tfrac{1}{2}\,r^2\,i\cos\theta + r\,A\sin^2\theta) + M_{Ozx}\left(\frac{r^2\,i\sin\theta}{2} - \frac{1}{2}\,r\,i\sin 2\,\theta + \frac{1}{2}\,r\,A\sin 2\,\theta\right)$$
$$+\,R\,a\left(\frac{1}{3} + \frac{i\,r^3}{3} - r^2\,i\cos\theta + r\,i\cos^2\theta + r\,A\sin^2\theta\right)$$
$$= M_T\,(\tfrac{1}{2}\,i\,r^2\sin\theta - \tfrac{1}{2}\,i\,r\sin 2\,\theta + \tfrac{1}{2}\,r\,A\sin 2\,\theta)$$

When angle θ is 0°, cantilevers a and b are parallel to each other. When θ is 90°, the conditions are identical for the right-hand cantilever. At angle θ equals 180°, the double cantilever becomes a beam with fixed ends, assuming that restraining moments would be acting. When these moments are zero, a simple beam will result.

A word of caution is necessary. These equations were developed in terms of distance a. Thus, when the angle is 180°, M_{Ozy} is expressed as $\frac{p\,a^2}{3}$ when both cantilevers of the same length carry uniformly distributed load. The conventional manner of indicating this moment is in terms of the entire length. Hence, if $L = 2\,a$, the bending moment M_{Ozy} will be equal to $\frac{p\,L^2}{12}$.

Effect of Shear

It is generally assumed by structural engineers that shear distortions produce minor additive forces to reactions, and a slight reduction to bending moments in conventional indeterminate structures, when the ratio of length to depth of beam is greater than four. Consequently, the shear effect is ignored, as a differential of a higher order. In order to investigate shear effects on a 90° double cantilever, two cases were studied, and equations for both are shown in Table 1.

The first case considered is one in which both cantilevers were loaded with uniformly distributed load p, the second, with a concentrated load P, at point O.

The general equation of work considering shear is

$$W = \int \frac{M^2\,ds}{2\,E\,I} + \int \frac{M^2\,ds}{2\,G\,J} + \int \frac{V^2\,ds}{2\,G\,A} \qquad (18)$$

The first two parts of Eq. 18 have been solved for a number of loading conditions. The third is a corrective factor due to shear, and is additive to Eq. c, Table 1. These corrective factors have been underscored in respective equations where they appear.

Settlements of Supports and Rotations of End Tangents

It may be necessary to determine the effects of settlement of end supports and rotations of end tangents. The three equations expressing vertical displacements and end rotations at point O, for a right-angle cantilever, are listed in Table 1. Their application is simple. The only precaution necessary is to determine consistent signs for displacements and angular rotations.

Practical Considerations

To simplify calculations, equations have been reduced into a series of dimensionless quantities. Terms on the left of the equality sign contain functions of elastic properties of the double cantilever and are invariants. On the right of the equality sign, the functions depend on the applied loads, and on the elastic and torsional properties of sections.

It has been found from actual calculations that these equations do not lend themselves readily to slide-rule calculations. In many instances, slide-rule calculations differed as much as 100% to 200% from true values. The reasons for these discrepancies are due to small differences of large quantities, which cannot be estimated closely on a 10-in. slide rule. In order to determine satisfactory values, calculations should be carried to three or four significant figures.

Variations in Moments and Shears

To show the effects of torsional stiffness on moments and shears, Table 2 has been prepared for a uniformly distributed load over both lengths of the right-angle cantilever, and also for a concentrated load P acting at point C. Both cantilevers have the same physical dimensions—that is, $A = B$, and variations are assumed from one to fifty.

TABLE 2.—Effects of Torsional Stiffness on Moments and Shears

$A\ (= B)$	(a) Concentrated Load, P ($c = a = b$ and $R = P/2$)		(b) Uniform Load, p ($a_2 = a = b = b_2$ and $R = p\,a$)	
	M_{Ozy}	M_{Ozz}	M_{Ozy}	M_{Ozz}
1	$3\,P\,a/8$	$-\,P\,a/8$	$5\,p\,a^2/12$	$-\,p\,a^2/12$
5	$11\,P\,a/24$	$-\,P\,a/24$	$17\,p\,a^2/36$	$-\,p\,a^2/36$
10	$21\,P\,a/44$	$-\,P\,a/44$	$32\,p\,a^2/66$	$-\,p\,a^2/66$
50	$101\,P\,a/204$	$-\,P\,a/204$	$152\,p\,a^2/306$	$-\,p\,a^2/306$

Applications to Continuous Structures

An isolated double cantilever is rarely encountered in practice. It is generally a segment of a continuous structure extending beyond points O and B. When ratios of A and B are reasonably large, it is convenient to treat resulting moments as moments due to cantilevers, and then distribute the resulting moments to the contiguous parts of the structure according to conventional methods of analysis. This procedure is not quite correct, but it should give workable results. Special situations may arise in the more important structures where beams at the ends of the cantilevers must be included in resisting both torsion and flexure. Formal methods of attack, using the theory of least work, would be quite involved, but not too difficult to solve. Because of the considerable length of time necessary to determine correct values, such procedures would be viewed unfavorably in most engineering offices.

Conclusions

It is unfortunate that the equations in this paper cannot be plotted to cover practical ranges met in daily practice. Some plots were attempted, but since they covered a very narrow range, they were omitted. It is important to observe that, when ratios of A, B, and C are small, torsional moments begin to have considerable importance in these structures, and should be taken into consideration. When the values of A, B, and C are large, torsional moments are of secondary importance as compared with bending moments.

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